

Enlargements of Filtrations and Applications

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Abstract

In this paper we review some old and new results about the enlargement of filtrations problem, as well as their applications to credit risk and insider trading problems. The enlargement of filtrations problem consists in the study of conditions under which a semimartingale remains a semimartingale when the filtration is enlarged, and, in such a case, how to find the Doob-Meyer decomposition. Filtrations may be enlarged in different ways. In this paper we consider initial and progressive filtration enlargements made by random variables and processes.

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1 Enlargement of filtrations

When considering a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, accounting for the information related to a given phenomenon, the *arrival* of new information induces the consideration of an *enlarged filtration* $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$, such that $\mathcal{G}_t \supseteq \mathcal{F}_t$, for each $t \geq 0$. More specifically, one considers the filtration $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ defined as $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t \supseteq \mathcal{F}_t$, $t \geq 0$, where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ is assumed to represent the new information. Traditionally, when \mathbb{H} is such that $\mathcal{H}_t = \sigma(L)$, for $t \geq 0$, and for some random variable L , then it is said that \mathbb{G} is an *initial enlargement* of the filtration \mathbb{F} . Otherwise, it is said that \mathbb{G} is a *progressive enlargement* of the filtration \mathbb{F} .

Some natural questions arise in a filtration enlargement setting. For instance, *In which cases an \mathbb{F} -semimartingale remains a semimartingale in the enlarged filtration \mathbb{G} ?*, and *How can we compute \mathbb{G} -Doob-Meyer decompositions in function of \mathbb{F} and \mathbb{H} ?* According to (1), the beginnings of the theory of enlargement

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of filtrations may be traced back to K. Itô, P.A. Mayer, and D. Williams who in the late seventies, independently and separately from each other, posed similar questions. So far, the study of enlargement of filtrations has been devoted mainly to cases as $(\mathcal{H}_t = \sigma(\tau))_{t \geq 0}$ and $(\mathcal{H}_t = \mathbf{1}_{\{\tau \leq t\}})_{t \geq 0}$, where τ is some stopping time. One of the drawbacks of the present approaches is that some rather restrictive or unrealistic assumptions has to be made on the stopping time in order to apply the approach. For instance, in credit risk theory, τ usually represents the *default time* of some contract. In order to preclude arbitrage, τ is assumed to be either *initial time* or *honest time*¹. On the other hand, only a few studies has been developed in the general setting. For instance, it can be considered the case² when $\mathbb{H} = (\mathcal{H}_t = \sigma(J_t))_{t \geq 0}$, for $(J_t = \inf_{s \geq t} X_s)_{t \geq 0}$, being $(X_t)_{t \geq 0}$ a 3-dimensional Bessel process. Nevertheless, this case can be reduced in fact to a case with random times, taking into account that

$$\{J_t < a\} = \{t < \Lambda_a\},$$

where $\Lambda_a = \sup\{t, X_t = a\}$.

In this paper we approach the problem of initial enlargement of filtrations by considering a general \mathcal{F} -measurable random variable L (as opposed to a stopping time) and looking into its law conditioned to the -not necessarily Brownian- filtration \mathbb{F} . Regarding to the first question, we give a condition under which \mathbb{F} -semimartingales remain semimartingales under the enlarged filtration \mathbb{G} . Regarding to the second question, we study conditions under which it is possible to obtain \mathbb{G} -Doob-Meyer decompositions as well as explicit expressions for the compensator. For progressive enlargement of filtrations, we present the example that motivated this paper and which represents an enlargement done by an \mathbb{H} that is not induced by a stopping time. Finally, we present applications of the enlargement of filtration theory in the field of mathematical finance, specifically in credit risk theory and insider trading.

1.1 Initial enlargement of filtrations

Consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration \mathbb{F} is assumed to satisfy the usual conditions. Let L be an \mathcal{F} -measurable random variable with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $T > 0$ denote a time horizon, and define $\mathcal{G}_t := \cap_{T \geq s > t} (\mathcal{F}_s \vee \sigma(L))$ and $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$. Notice that defining \mathcal{G}_t as $\cap_{T \geq s > t} (\mathcal{F}_s \vee \sigma(L))$ assures the right-continuity of the filtration \mathbb{G} , and therefore, that \mathbb{G} satisfy the usual conditions.

Condition A. For all t , there exists a σ -finite measure η_t in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Q_t(\omega, \cdot) \ll \eta_t$ where

¹See section 2.1 for more details.

²See Section 1.2.2 in (1).

$Q_t(\omega, dx)$ is a regular version of $L|\mathcal{F}_t$.

Notice that **Condition A** is satisfied in the case that L takes a countable number of values and the case when L is independent of \mathcal{F}_∞ , just taking η_t the law of L . An example where **Condition A** is not satisfied is the following example:

Example 1 Let L be the n -th jump of a Poisson process $(N_t)_{t \in [0, T]}$ with intensity λ and \mathbb{F} the filtration generated by $(N_t)_{t \in [0, T]}$, then

$$\mathbb{P}\{L > x | \mathcal{F}_t\} = \mathbf{1}_{\{N_x < n, N_t \geq n\}} + \mathbf{1}_{\{N_t < n\}} \int_{(x-t)_+}^{\infty} \frac{\lambda e^{-\lambda u} (\lambda u)^{n-N_t-1}}{(n-N_t-1)!} du,$$

then the conditional probability cannot be dominated by a non random measure.

Theorem 2 Under **Condition A** any X , \mathbb{F} -semimartingale is a \mathbb{G} -semimartingale.

Proof. The proof is based in the characterization of semimartingales of Bichteler-Dellacherie-Mokobodzki: Let X be a càdlàg process, adapted to a filtration \mathbb{H} define the class of predictable processes

$$\xi_{\rho, t} := \left\{ f = \sum_{i=1}^{n-1} f_i \mathbf{1}_{(s_i, s_{i+1}]} , 0 = s_0 < \dots s_n \leq t, f_i \in H_{s_i-}, |f_i| < \rho \right\}$$

define

$$\int_0^t f_s dX_s := \sum_{i=1}^{n-1} f_i (X_{s_{i+1}} - X_{s_i})$$

and, for $Z \in L^1$

$$\alpha_{\rho, t}^X(Z, \mathbb{H}) = \sup_{f \in \xi_{\rho, t}} \mathbb{E} \left[|Z| \left(1 \wedge \int_0^t f_s dX_s \right) \right],$$

then,

$$\begin{aligned} X &\in S(\mathbb{H}) \iff \lim_{\rho \rightarrow 0} \alpha_{\rho, t}^X(1, \mathbb{H}) = 0 \\ X &\in S(\mathbb{H}) \implies \lim_{\rho \rightarrow 0} \alpha_{\rho, t}^X(Z, \mathbb{H}) = 0. \end{aligned}$$

This result together with the fact that any f , \mathcal{G}_i -measurable function, can be written as $f = g(\omega)^{L(\omega)}$, where $g(\omega)^x$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_i$ -measurable, allows to get the result. ■

Proposition 3 **Condition A** is equivalent to $Q_t(\omega, dx) \ll \eta(dx)$ where η is the law of L .

Proof. By **Condition A** we have that $Q_t(\omega, dx) = q_t^x(\omega) \eta_t(dx)$, where $q_t^x(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ -measurable then we can write $Q_t(\omega, dx) = \hat{q}_t^x(\omega) \eta(dx)$ with $\hat{q}_t^x(\omega) = \frac{q_t^x(\omega)}{\mathbb{E}(q_t^x(\omega))}$. ■

Proposition 4 Under **Condition A** there exists $q_t^x(\omega) \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ -measurable such that

$$Q_t(\omega, dx) = q_t^x(\omega) \eta(dx) \quad (1)$$

and, for fixed x , $(q_t^x)_{t \in [0, T]}$ is an \mathbb{F} -martingale.

Proof. See (2) Lemma 1.8. ■

Theorem 5 Let $(M_t)_{t \in [0, T]}$ be a continuous local \mathbb{F} -martingale and $k_t^x(\omega)$ such that

$$\langle q^x, M \rangle_t = \int_0^t k_s^x q_{s-}^x d\langle M, M \rangle_s$$

then

$$M - \int_0^\cdot k_s^L d\langle M, M \rangle_s,$$

is a \mathbb{G} -martingale.

Proof. Except for a localization procedure (see details in (2) Theorem 2.1) the proof is the following: let $s < t$, $Z \in \mathcal{F}_s$, and g be abounded Borel function, then

$$\begin{aligned} \mathbb{E}[Zg(L)(M_t - M_s)] &= \mathbb{E}[\mathbb{E}[Zg(L)(M_t - M_s)|\mathcal{F}_t]] \\ &= \mathbb{E}[Z(M_t - M_s)\mathbb{E}[g(L)|\mathcal{F}_t]] \\ &= \int_{\mathbb{R}} g(x) \eta(dx) \mathbb{E}[Z(M_t - M_s)q_t^x] \\ &= \int_{\mathbb{R}} g(x) \eta(dx) \mathbb{E}[Z(M_t q_t^x - M_s q_s^x)] \\ &= \int_{\mathbb{R}} g(x) \eta(dx) \mathbb{E}[Z(\langle M, q^x \rangle_t - \langle M, q^x \rangle_s)] \\ &= \int_{\mathbb{R}} g(x) \eta(dx) \mathbb{E}\left[Z\left(\int_s^t k_u^x q_{u-}^x d\langle M, M \rangle_u\right)\right] \\ &= \mathbb{E}\left[Zg(L)\left(\int_s^t k_u^x q_{u-}^x d\langle M, M \rangle_u\right)\right] \end{aligned}$$

■

Example 6 Let $T = 1$, and take $(M_t := B_t, t \in [0, T])$ where $(B_t)_{t \in [0, T]}$ is a standard Brownian motion, and take $L := B_T$. It follows easily from (1) that

$$q_t^x(\omega) \sim \frac{1}{(T-t)^{1/2}} \exp\left(-\frac{1}{2(T-t)}(B_t(\omega) - x)^2 + \frac{x^2}{2}\right).$$

Applying Itô's formula we get

$$dq_t^x = q_t^x \frac{x - B_t}{T - t} dB_t,$$

hence $k_t^x = \frac{x - B_t}{T - t}$ and

$$B. - \int_0^\cdot \frac{B_T - B_s}{T - s} ds$$

is a $\mathbb{G} := \mathbb{F}^B \vee \sigma(B_T)$ martingale. Note that, by the Lévy theorem, $B. - \int_0^\cdot \frac{B_T - B_s}{T - s} ds$ is a (standard) \mathbb{G} -Brownian motion and, since B_T is \mathcal{G}_0 -measurable, it is independent of $(W_t)_{t \in [0, T]}$.

Example 7 Note that if the filtration \mathbb{F} is that generated by a Brownian motion, $(B_t)_{t \in [0, T]}$, then for any \mathbb{F} -martingale $(M_t)_{t \in [0, T]}$ we have

$$dM_t = \sigma_t dB_t$$

and so

$$d\langle M, M \rangle_t = \sigma_t^2 dt.$$

Also, assuming that

$$q_t^x(\omega) = h_t^x(B_t)$$

and $h \in C^{1,2}$ we will have that

$$dq_t^x = \partial h_t^x(B_t) dB_t,$$

and

$$k_t^x = \frac{\partial \log h_t^x(B_t)}{\sigma_t}.$$

Example 8 The previous example can be generalized as follows: let $(Y_t)_{t \in [0, T]}$ be the following Brownian semimartingale

$$Y_t = Y_0 + \int_0^t \sigma(Y_s) dB_s + \int_0^t b(Y_s) ds,$$

and assume that

$$Y_T | \mathcal{F}_t \sim \pi(T - t, Y_t, x) dx,$$

with π smooth. We know that $(\pi(T - t, Y_t, x))_t$ is an \mathbb{F} -martingale, then

$$d\pi(T - t, Y_t, x) = \frac{\partial \pi}{\partial y}(T - t, Y_t, x) \sigma(Y_t) dB_t$$

and by the Jacod theorem

$$\int_0^\cdot \sigma(Y_s) dB_s - \int_0^\cdot \frac{\partial \log \pi}{\partial y}(T-s, Y_s, Y_T) \sigma^2(Y_s) ds$$

is an $\mathbb{F} \vee \sigma(Y_T)$ -martingale, and we can write

$$Y_\cdot = Y_0 + \int_0^\cdot \sigma(Y_s) d\tilde{B}_s + \int_0^\cdot b(Y_s) ds + \int_0^\cdot \frac{\partial \log \pi}{\partial y}(T-s, Y_s, Y_T) \sigma^2(Y_s) ds,$$

where $(\tilde{B}_t)_{t \in [0, T]}$ is an $\mathbb{F} \vee \sigma(Y_T)$ -Brownian motion.

We have also a similar result for locally bounded martingales.

Theorem 9 *Let $(M_t)_{t \in [0, T]}$ be an \mathbb{F} -local martingale locally bounded. Then, there exist $k_t^x(\omega)$ such that*

$$\langle q^x, M \rangle_\cdot = \int_0^\cdot k_s^x q_{s-}^x d\langle M, M \rangle_s,$$

and

$$M_\cdot - \int_0^\cdot k_s^L d\langle M, M \rangle_s$$

is a \mathbb{G} -martingale.

Suppose that any \mathbb{F} -local martingale admits a representation of the form

$$M_t = M_0 + \sum_{(n)} \int_0^t K_s^n dX_s^n + \int_0^t \int_{\mathbb{R}} W(\omega, x, s) (Q(\omega, dx, ds) - \nu(\omega, dx, ds))$$

where (X^n) are continuous local martingales pairwise orthogonal, assume that q^x admits the representation

$$q_t^x = q_0^x + \sum_{(n)} \int_0^t k_s^{n,x} q_{s-}^x dX_s^n + \int_0^t \int_{\mathbb{R}} q_{s-}^x U_s^x (Q(\cdot, dx, ds) - \nu(\cdot, dx, ds)),$$

then

$$M_t - \sum_{(n)} \int_0^t K_s^n k_s^{n,L} d\langle X^n, X^n \rangle_s - \int_0^t \int_{\mathbb{R}} W(\cdot, x, s) U_s^L \nu(\cdot, dx, ds)$$

is a \mathbb{G} -martingale with continuous part,

$$M_0 + \sum_{(n)} \int_0^t K_s^n dX_s^n - \sum_{(n)} \int_0^t K_s^n k_s^{n,L} d\langle X^n, X^n \rangle_s$$

and jump part

$$\int_0^t \int_{\mathbb{R}} W(\omega, x, s) (Q(\omega, dx, ds) - (1 + U_s^L) \nu(\omega, dx, ds)).$$

Example 10 Consider the Poisson process $(N_t)_{t \in [0,1]}$ of intensity λ , as well as the filtration $\mathbb{F} := \mathbb{F}^N$ generated by it. Let

$$M_t = N_t - \lambda t, \quad t \in [0, 1],$$

and $L = N_1$, then

$$\begin{aligned} Q_t(\cdot, dk) &= \mathbb{P}\{N_1 = k | \mathcal{F}_t\} = \mathbb{P}\{N_{1-t} = k - N_t\} \\ &= e^{-\lambda(1-t)} \frac{(\lambda(1-t))^{k-N_t}}{(k-N_t)!}, \end{aligned}$$

and

$$\begin{aligned} q_t^k &= \frac{e^{-\lambda(1-t)} \frac{(\lambda(1-t))^{k-N_t}}{(k-N_t)!}}{e^{-\lambda} \frac{\lambda^k}{k!}} \\ &= \frac{e^{\lambda t} \lambda^{-N_t} (1-t)^{k-N_t} k!}{(k-N_t)!}. \end{aligned}$$

Now, if there is a jump at t :

$$U_t^k = \frac{q_t^k - q_{t-}^k}{q_{t-}^k} = \frac{k - N_{t-}}{\lambda(1-t)} - 1,$$

so

$$1 + U_t^L = \frac{N_1 - N_{t-}}{\lambda(1-t)},$$

and

$$N_t - \int_0^t \frac{N_1 - N_s}{1-s} ds, 0 \leq t < 1$$

is a \mathbb{G} -martingale.

Remark 11 A more general result, concerning Lévy processes, can be obtained by using the characteristic function instead of the conditional density. Let $(Z_t)_{t \geq 0}$ be a Lévy process with characteristic function

$$\mathbb{E}[e^{i\theta Z_t}] = e^{t\psi(\theta)}.$$

Let $0 \leq s \leq u \leq t \leq T$. Then, in virtue of the independence of the increments of $(Z_t)_{t \geq 0}$, we have that the

following chain of equations

$$\begin{aligned}
\mathbb{E}[e^{i\theta Z_T}(Z_t - Z_u)h_s] &= \mathbb{E}[e^{i\theta(Z_T - Z_t)}]\mathbb{E}[e^{i\theta(Z_t - Z_u)}(Z_t - Z_u)]\mathbb{E}[e^{i\theta Z_u}h_s] \\
&= \mathbb{E}[e^{i\theta(Z_T - Z_t)}]\left(\frac{1}{i}\mathbb{E}[e^{i\theta(Z_t - Z_u)}]\partial_\theta \log \mathbb{E}[e^{i\theta(Z_t - Z_u)}]\right)\mathbb{E}[e^{i\theta Z_u}h_s] \\
&= \frac{1}{i}\mathbb{E}[e^{i\theta Z_T}h_s]\partial_\theta \log \mathbb{E}[e^{i\theta(Z_t - Z_u)}] \\
&= \frac{1}{i}(t - u)\mathbb{E}[e^{i\theta Z_T}h_s]\partial_\theta \psi(\theta).
\end{aligned}$$

And, consequently,

$$\mathbb{E}\left[e^{i\theta Z_T}h_s\frac{Z_t - Z_u}{t - u}\right] = \frac{1}{i}\mathbb{E}[e^{i\theta Z_T}h_s]\partial_\theta \psi(\theta).$$

Since the right hand side of previous equation does not depend on t , we have

$$\mathbb{E}\left[e^{i\theta Z_T}h_s\frac{Z_T - Z_u}{T - u}\right] = \frac{1}{i}\mathbb{E}[e^{i\theta Z_T}h_s]\partial_\theta \psi(\theta).$$

Thus, by integrating with respect u and applying Fubini we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta Z_T}h_s\int_s^t\frac{Z_T - Z_u}{T - u}du\right] &= \frac{1}{i}(t - s)\mathbb{E}[e^{i\theta Z_T}h_s]\partial_\theta \psi(\theta) \\
&= \mathbb{E}[e^{i\theta Z_T}h_s(Z_t - Z_s)].
\end{aligned}$$

In certain non-homogeneous cases we can use a similar argument. Assume that

$$\mathbb{E}[e^{i\theta(Z_t - Z_u)}] = e^{g(t, u)\psi(\theta)},$$

then

$$\mathbb{E}\left[e^{i\theta Z_T}h_s\int_s^t\frac{Z_T - Z_u}{g(T, u)}(-1)\partial_u g(t, u)du\right] = \mathbb{E}[e^{i\theta Z_T}h_s(Z_t - Z_s)],$$

provided that $\int_s^t\frac{Z_T - Z_u}{g(T, u)}(-1)\partial_u g(t, u)du$ is well defined.

Condition A is a sufficient condition that allow us to find the Doob decomposition in the enlarged filtration.

However we have the following proposition that allows us to obtain the Doob-Meyer decomposition without requiring **Condition A**:

Proposition 12 *With the notations above, assume that there exist $\alpha_t^x(\omega)$ such that*

$$\left\langle \int_a^\infty Q_t(\cdot, dx), M \right\rangle = \int_0^t \int_a^\infty \alpha_s^x Q_{s-}(\cdot, dx) d\langle M, M \rangle_s, \text{ for all } a \in \mathbb{R},$$

then

$$M. - \int_0^\cdot \alpha_s^x d\langle M, M \rangle_s,$$

is a \mathbb{G} -martingale.

Proof. For every $Z \in \mathcal{F}_s$ we have that

$$\begin{aligned} \mathbb{E}[Z \mathbf{1}_{\{L > a\}}(M_t - M_s)] &= \mathbb{E}[\mathbb{E}[Z \mathbf{1}_{\{L > a\}}(M_t - M_s) | \mathcal{F}_t]] \\ &= \mathbb{E}[Z(M_t - M_s) \mathbb{E}[\mathbf{1}_{\{L > a\}} | \mathcal{F}_t]] \\ &= \mathbb{E}\left[Z(M_t - M_s) \int_a^\infty Q_t(\omega, dx)\right] \\ &= \mathbb{E}\left[Z\left(M_t \int_a^\infty Q_t(\omega, dx) - M_s \int_a^\infty Q_s(\omega, dx)\right)\right] \\ &= \mathbb{E}\left[Z\left(\langle M, \int_a^\infty Q_\cdot(\omega, dx) \rangle_t - \langle M, \int_a^\infty Q_\cdot(\omega, dx) \rangle_s\right)\right] \\ &= \mathbb{E}\left[Z \int_s^t \int_a^\infty \alpha_u^x Q_{u-}(\omega, dx) d\langle M, M \rangle_u\right] \\ &= \mathbb{E}\left[Z \mathbf{1}_{\{L > a\}} \int_s^t \alpha_u^L d\langle M, M \rangle_u\right] \end{aligned}$$

■

Example 13 Let $(B_t)_{t \geq 0}$ be a Brownian motion and take $\tau = \inf\{t > 0, B_t = -1\}$. It is well known that

$$\mathbb{P}\{\tau \leq s | \mathcal{F}_t\} = 2\Phi\left(-\frac{1+B_t}{\sqrt{s-t}}\right) \mathbf{1}_{\{\tau \wedge s > t\}} + \mathbf{1}_{\{s < \tau \wedge t\}},$$

where Φ is the c.d.f. of a standard normal distribution. Then in $t < s \wedge \tau$ we have, by Itô's formula,

$$\mathbb{P}\{\tau \leq s | \mathcal{F}_t\} = 2\Phi\left(-\frac{1}{\sqrt{s}}\right) + \sqrt{\frac{2}{\pi}} \int_0^t \frac{1}{\sqrt{s-u}} e^{-\frac{(1+B_u)^2}{2(s-u)}} dB_u,$$

so

$$d\langle \mathbb{P}\{\tau \leq s | \mathcal{F}_\cdot\}, B \rangle_t = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{s-t}} e^{-\frac{(1+B_t)^2}{2(s-t)}} dt,$$

and

$$\begin{aligned} &\alpha_t^s Q_t(\cdot, ds) \\ &= \frac{\partial}{\partial s} \left(\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{s-t}} e^{-\frac{(1+B_t)^2}{2(s-t)}} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{(s-t)^3}} - \frac{(1+B_t)^2}{\sqrt{(s-t)^5}} \right) e^{-\frac{(1+B_t)^2}{2(s-t)}}, \end{aligned}$$

finally

$$Q_t(\cdot, ds) = \frac{\partial}{\partial s} \mathbb{P}\{\tau \leq s | \mathcal{F}_t\} = \frac{e^{-\frac{(1+B_t)^2}{2(s-t)}}}{\sqrt{2\pi}\sqrt{(s-t)^3}} (1+B_t),$$

and

$$\alpha_t^s = \frac{\frac{\partial}{\partial s} \left(\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{s-t}} e^{-\frac{(1+B_t)^2}{2(s-t)}} \right)}{\frac{\partial}{\partial s} \mathbb{P}\{\tau \leq s | \mathcal{F}_t\}} = \frac{1}{1+B_t} - \frac{1+B_t}{s-t}.$$

Consequently

$$B_t - \int_0^{t \wedge \tau} \left(\frac{1}{1+B_s} - \frac{1+B_s}{\tau-s} \right) ds, \quad t \geq 0,$$

is a \mathbb{G} -martingale.

Proposition 14 *If we have a random measure $P_t^{(1)}(\omega, dx)$ and a finite deterministic measure $m(dt)$ such that*

$$\mathbb{E}[Zg(L)(M_t - M_s)] = \mathbb{E} \left[\int_s^t \int_{\mathbb{R}} Zg(x) P_t^{(1)}(\omega, dx) m(dt) \right]$$

and $P_t^{(1)}(\omega, dx) = \alpha_t^x(\omega) Q_t(\omega, dx)$ then

$$M_t - \int_0^t \alpha_s^L m(ds)$$

is a \mathbb{G} -martingale.

Proof.

$$\begin{aligned} \mathbb{E}[Zg(L)(M_t - M_s)] &= \mathbb{E} \left[\int_s^t \int_{\mathbb{R}} Zg(x) P_u^{(1)}(\omega, dx) m(du) \right] \\ &= \mathbb{E} \left[\int_s^t \int_{\mathbb{R}} Zg(x) \alpha_u^x(\omega) Q_u(\omega, dx) m(du) \right] \\ &= \mathbb{E} \left[\int_s^t Z \mathbb{E}[g(L) \alpha_u^L | \mathcal{F}_t] m(du) \right] \\ &= \mathbb{E} \left[\int_s^t Zg(L) \alpha_u^L m(du) \right] \end{aligned}$$

■

This expression is more appropriate to treat cases like Example 1. Suppose that L is a random time and that there exist $P_u^{(1)}(\omega, dx)$ and $P_u^{(2)}(\omega, dx)$ such that

$$\begin{aligned} \mathbb{E}[Zg(L) \mathbf{1}_{\{L < s\}}(M_t - M_s)] &= \mathbb{E} \left[\int_s^t \int_0^s Zg(x) P_u^{(1)}(\omega, dx) m_1(du) \right] \\ \mathbb{E}[Zg(L) \mathbf{1}_{\{L > t\}}(M_t - M_s)] &= \mathbb{E} \left[\int_s^t \int_t^\infty Zg(x) P_u^{(2)}(\omega, dx) m_2(du) \right], \end{aligned}$$

and that $P_u^{(i)}(\omega, dx) = \alpha_u^{x(i)}(\omega)Q_t(\omega, dx)$, $i = 1, 2$, then it is easy to see, by decomposing $M_t - M_s$ as sum of increments, that

$$M_t - \int_0^t \mathbf{1}_{\{L < u\}} \alpha_u^{L(1)} m_1(du) - \int_0^t \mathbf{1}_{\{L > u\}} \alpha_u^{L(2)} m_2(du) - \Delta M_L \mathbf{1}_{\{L \leq t\}}, \quad t \in [0, T],$$

is a \mathbb{G} -martingale.

Example 15 Consider Example 1. $L \equiv T_n$,

$$\begin{aligned} \mathbb{E}[Zg(T_n)\mathbf{1}_{\{T_n < s\}}(N_t - N_s)] &= \mathbb{E}[(N_t - N_s)]\mathbb{E}[Z\mathbb{E}[g(T_n)\mathbf{1}_{\{T_n < s\}}]] \\ &= \mathbb{E}\left[\int_s^t \int_0^s \lambda Zg(x)Q_t(\omega, dx)du\right] \end{aligned}$$

so,

$$P_t^{(1)}(\omega, dx) = \lambda Q_t(\omega, dx).$$

and

$$\begin{aligned} &\mathbb{E}[Zg(T_n)\mathbf{1}_{\{T_n > t\}}(N_t - N_s)] \\ &= \mathbb{E}[(N_t - N_s)Z\mathbb{E}[g(T_n)\mathbf{1}_{\{T_n > t\}}|\mathcal{F}_t]] \\ &= \mathbb{E}\left[(N_t - N_s)Z \int_t^\infty \frac{\lambda e^{-\lambda(x-t)} (\lambda(x-t))^{n-N_t-1}}{(n-N_t-1)!} g(x)dx\right] \\ &= \mathbb{E}\left[(N_t - N_s)Z \int_t^\infty \frac{\lambda e^{-\lambda(x-t)} (\lambda(x-t))^{n-(N_t-N_s)-1-N_s}}{(n-(N_t-N_s)-1-N_s)!} g(x)dx\right] \\ &= \mathbb{E}\left[Z \sum_{k=1}^{n-1-N_s} \int_t^\infty \frac{\lambda e^{-\lambda(x-t)} (\lambda(x-t))^{n-k-1-N_s}}{(n-k-1-N_s)!} \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{(k-1)!} g(x)dx\right] \\ &= \lambda(t-s)\mathbb{E}\left[Z \sum_{k=0}^{n-2-N_s} \int_t^\infty \frac{\lambda e^{-\lambda(x-t)} (\lambda(x-t))^{n-k-2-N_s}}{(n-k-2-N_s)!} \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!} g(x)dx\right] \\ &= (t-s)\mathbb{E}\left[Z\mathbb{E}\left[\frac{n-N_t-1}{T_n-t} g(T_n)\mathbf{1}_{\{T_n > t\}} \middle| \mathcal{F}_t\right]\right] \\ &= \mathbb{E}\left[Z \int_s^t \int_t^\infty \frac{n-N_t-1}{x-t} g(x)Q_t(\omega, dx)dt\right], \end{aligned}$$

therefore

$$P_t^{(2)}(\omega, dx) = \frac{n-N_t-1}{x-t} Q_t(\omega, dx).$$

Consequently

$$N_t - \lambda(t - T_n \wedge t) - \int_0^{T_n \wedge t} \frac{n - N_u - 1}{T_n - u} du - \mathbf{1}_{\{T_n \leq t\}}, \quad t \geq 0,$$

is a \mathbb{G} -martingale. We also can write

$$N_t - \lambda(t - T_n \wedge t) - \int_0^{T_n \wedge t} \frac{N_{T_n-} - N_u}{T_n - u} du - \mathbf{1}_{\{T_n \leq t\}},$$

we can compare to Example 10, note also that there is not jump at T_n . We can use Proposition 12 instead.

If we take $t < T_n \wedge s$

$$\begin{aligned} \mathbb{P}\{T_n > s | \mathcal{F}_t\} &= \int_{(s-t)_+}^{\infty} \frac{\lambda e^{-\lambda u} (\lambda u)^{n-N_t-1}}{(n-N_t-1)!} du \\ &= \int_s^{\infty} \frac{\lambda e^{-\lambda(u-t)} (\lambda(u-t))^{n-N_t-1}}{(n-N_t-1)!} du = \int_s^{\infty} Q_t(\cdot, du), \end{aligned}$$

then, with $(M_t := N_t - \lambda t, t \in [0, T])$ we have

$$\left\langle \int_s^{\infty} Q_t(\cdot, du), M \right\rangle = \int_s^{\infty} \frac{\lambda e^{-\lambda(u-t)} (\lambda(u-t))^{n-N_t}}{(n-N_t)!} \left(\frac{n-1-N_{t-}}{\lambda(u-t)} - 1 \right) \lambda dt,$$

therefore

$$N_t - \int_0^t \frac{N_{T_n-} - N_u}{T_n - u} du, \quad t < T_n,$$

is a \mathbb{G} -martingale. Note that, by using this proposition, we cannot extend the \mathbb{G} -martingale to values of $t \geq T_n$.

1.2 Progressive enlargement of filtrations

In the progressive enlargement of filtrations one consider a filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$ with $\mathcal{G}_t = \cap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s)$, where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ is another filtration. The case where $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau \leq t\}})$ with τ a random time has been extensively studied, see for instance (3)-, (4) or (1), among others. However, as mentioned in the introduction, few studies has been developed in the general setting. We present now an example of an exception, extract from (5), in which $\mathcal{H}_t = \sigma(L_t)$, $t \in [0, T]$, for $L_t = G(X, Y_t)$, where X is an \mathcal{F}_T -measurable random variable, $(Y_t)_{t \geq 0}$ is a process independent of \mathcal{F}_T , and G is a Borelian function. The following proposition gives a particular case of this situation, whereas in section §2.2 below, we give the applications context from which this example arose.

Proposition 16 Assume that $(B_t)_{t \in [0, T]}$ is a Brownian motion and take $\mathbb{F} := \mathbb{F}^B$. Let $(W_t)_{t \in [0, T]}$ be another Brownian motion independent of $(B_t)_{t \in [0, T]}$, and consider the process $V_t := B_t + \int_t^T \sigma_s dW_s$, with

$\int_t^T \sigma_s^2 ds < \infty$, for all $0 \leq t \leq T$. Then, provided that

$$\int_0^t \frac{T}{T + \int_s^T \sigma_u^2 du - s} ds < \infty,$$

we have that the Doob-Meyer decomposition of $(B_t)_{t \in [0, T]}$ in $\mathbb{F}^{B, V}$ is given by

$$B_t = \tilde{W}_t + \int_0^t \frac{V_s - B_s}{T + \int_s^T \sigma_u^2 du - s} ds, \quad 0 \leq t < T$$

where $(\tilde{W}_t)_{t \in [0, T]}$ is a Brownian motion but correlated with $(V_t)_{t \in [0, T]}$.

Proof. $(\tilde{W}_t)_{t \in [0, T]}$ is a centered Gaussian processes and for $0 \leq s \leq t < T$

$$\begin{aligned} \mathbb{E}[\tilde{W}_t \tilde{W}_s] &= \mathbb{E} \left[\left(B_t - \int_0^t \frac{V_u - B_u}{T + \int_u^T \sigma_v^2 dv - u} du \right) \left(B_s - \int_0^s \frac{V_u - B_u}{T + \int_u^T \sigma_v^2 dv - u} du \right) \right] \\ &= s - \mathbb{E} \left[B_t \int_0^s \frac{V_u - B_u}{T + \int_u^T \sigma_v^2 dv - u} du \right] - \mathbb{E} \left[B_s \int_0^t \frac{V_u - B_u}{T + \int_u^T \sigma_v^2 dv - u} du \right] \\ &\quad + \mathbb{E} \left[\int_0^t \frac{V_u - B_u}{T + \int_u^T \sigma_v^2 dv - u} du \int_0^s \frac{V_u - B_u}{T + \int_u^T \sigma_v^2 dv - u} du \right] \\ &= s - \int_0^s \frac{t - u}{T + \int_u^T \sigma_v^2 dv - u} du - \int_0^s \frac{s - u}{T + \int_u^T \sigma_v^2 dv - u} du \\ &\quad + 2\mathbb{E} \left[\int_0^s \left(\int_0^r \frac{(V_r - B_r)(V_u - B_u)}{(T + \int_r^T \sigma_v^2 dv - r)(T + \int_u^T \sigma_v^2 dv - u)} du \right) dr \right] \\ &\quad + \mathbb{E} \left[\int_s^t \left(\int_0^s \frac{(V_r - B_r)(V_u - B_u)}{(T + \int_r^T \sigma_v^2 dv - r)(T + \int_u^T \sigma_v^2 dv - u)} du \right) dr \right] \\ &= s - \int_0^s \frac{s + t - 2u}{T + \int_u^T \sigma_v^2 dv - u} du + 2 \int_0^s \frac{s - u}{T + \int_u^T \sigma_v^2 dv - u} du + \int_0^s \frac{t - s}{T + \int_u^T \sigma_v^2 dv - u} du \\ &= s. \end{aligned}$$

On the other hand, for $t \geq s$

$$\mathbb{E}[\tilde{W}_t V_s] = s - \int_0^s \frac{T + \int_s^T \sigma_v^2 dv - u}{T + \int_u^T \sigma_v^2 dv - u} du > 0,$$

provided that σ_v is not identically null (a.e.). ■

Remark 17 It is important to note that contrarily to the case of initial enlargement, the innovation process $(\tilde{W}_t)_{t \in [0, T]}$ is not necessarily independent of the additional information. Then this fact makes the application

of enlargement of filtrations in our framework more involved. In other words, in most models, it is assumed that the privilege information $(V_t)_{t \in [0, T]}$ is independent of the demand process of liquidity traders $(\tilde{W}_t)_{t \in [0, T]}$. Consequently, the previous proposition cannot be used with these models. Instead, we have to look for processes such that their Doob-Meyer decomposition is of the form

$$X_t = \tilde{W}_t + \int_0^t \theta(V_t; X_u, 0 \leq u \leq s) ds, \quad 0 \leq t \leq T,$$

where $(\tilde{W}_t)_{t \in [0, T]}$ and $(V_t)_{t \in [0, T]}$ are independent.

Now consider the case when $\mathcal{H}_t = \sigma(V_t)$ for

$$V_t = V_0 + \int_0^t \sigma_s dW_s^1, \quad t \geq 0,$$

where σ_s is a deterministic function, V_0 is a zero mean normal r.v., and $(W_t^1, W_t^2)_{t \in [0, T]}$ is a 2-dimensional Brownian motion independent of V_0 . We have the following proposition:

Proposition 18 *Assume that $\text{Var}(V_T) = 1$ and that*

$$\int_0^t \frac{ds}{\text{Var}(V_s) - s} < \infty \text{ for all } 0 \leq t < T.$$

Then

$$B_t = W_t^2 + \int_0^t \frac{V_s - B_s}{\text{Var}(V_s) - s} ds, \quad 0 \leq t \leq T,$$

is a Brownian motion with $B_T = V_T$.

Proof. Let us denote $v_x := \text{Var}(V_x)$, for $0 \leq x \leq T$. We have

$$B_t = \int_0^t \exp\left(-\int_y^t \frac{1}{v_x - x} dx\right) dW_y^2 + \int_0^t \exp\left(-\int_u^t \frac{1}{v_x - x} dx\right) \frac{V_y}{v_y - y} dy,$$

so $(B_t)_{t \in [0, T]}$ is a centered Gaussian process, and for $s \leq t < T$,

$$\begin{aligned}
\mathbb{E}[B_t B_s] &= \exp\left(-\int_s^t \frac{1}{v_x - x} dx\right) \\
&\quad + \mathbb{E}\left[\int_0^t \int_0^s \exp\left(-\int_y^t \frac{1}{v_x - x} dx\right) \exp\left(-\int_z^s \frac{1}{v_x - x} dx\right) \frac{V_y V_z}{(v_y - y)(v_z - z)} dy dz\right] \\
&= \exp\left(-\int_s^t \frac{1}{v_x - x} dx\right) \int_0^s \exp\left(-2 \int_y^s \frac{1}{v_x - x} dx\right) dy \\
&\quad + \int_s^t \int_0^s \exp\left(-\int_y^t \frac{1}{v_x - x} dx\right) \exp\left(-\int_z^s \frac{1}{v_x - x} dx\right) \frac{v_z}{(v_y - y)(v_z - z)} dy dz \\
&\quad + 2 \int_0^s \int_0^y \exp\left(-\int_u^t \frac{1}{v_x - x} dx\right) \exp\left(-\int_z^s \frac{1}{v_x - x} dx\right) \frac{v_z}{(v_y - y)(v_z - z)} dy.
\end{aligned}$$

Then, since

$$\int_0^s \exp\left(-\int_z^s \frac{1}{v_x - x} dx\right) \frac{v_z}{v_z - z} dz = s,$$

and

$$2 \int_0^s \exp\left(-2 \int_z^s \frac{1}{v_x - x} dx\right) \frac{v_z}{v_z - z} dz = 2s + \int_0^s \exp\left(-2 \int_y^s \frac{1}{v_x - x} dx\right) dy$$

we obtain that $\mathbb{E}[B_t B_s] = s$. So for $0 \leq t < T$ we have that $(B_t)_{t \in [0, T]}$ is a standard Brownian motion. On the other hand

$$\begin{aligned}
\mathbb{E}[B_t V_t] &= \mathbb{E}\left[\int_0^t \exp\left(-\int_y^t \frac{1}{v_x - x} dx\right) \frac{V_y V_t}{v_y - y} dy\right] \\
&= \int_0^t \exp\left(-\int_y^t \frac{1}{v_x - x} dx\right) \frac{v_y}{v_y - y} dy \\
&= t,
\end{aligned}$$

therefore

$$\begin{aligned}
\mathbb{E}[(B_t - V_t)^2] &= \mathbb{E}[B_t^2] + \mathbb{E}[V_t^2] - 2\mathbb{E}[B_t V_t] \\
&= t + v_t - 2t = v_t - t,
\end{aligned}$$

and, since by hypothesis $v_T = 1$, this means that

$$\lim_{t \rightarrow T} B_t \stackrel{L^2}{=} V_T,$$

then for all $0 \leq t < T$

$$\mathbb{E} \left[\int_0^t \frac{|V_s - B_s|}{v_s - s} ds \right] < \int_0^t \frac{\mathbb{E}[(V_s - B_s)^2]^{\frac{1}{2}}}{v_s - s} ds = \int_0^t \sqrt{v_s - s} ds < \sqrt{2},$$

and this implies, by the monotone convergence theorem, that

$$\lim_{t \rightarrow T} \int_0^t \frac{|V_s - B_s|}{v_s - s} ds = \int_0^T \frac{|V_s - B_s|}{v_s - s} ds < \infty$$

and that $B_T = \lim_{t \rightarrow T} B_t$ is well defined. Now, we have, by the uniqueness of the limit in probability, that $V_T = B_T$ a.s. ■

2 Applications of Enlargement of Filtrations to Mathematical Finance

In mathematical finance, the enlargement of filtration theory may be applied to credit risk and to insider trading. In credit risk, the original filtration may be thought to represent the information related to defaultable-free assets in the market. We then construct the enlarged filtration by introducing the information related to a defaultable prone asset of interest. On the other hand, for insider trading, the reference filtration is that of a regular market agent. This filtration is enlarged by the *insider* agent who has privileged information about, for instance, the future value of said asset. In the following subsections we give more details about said applications.

2.1 Applications to credit risk theory

The main objective of quantitative models of credit risk is to provide ways to price and hedge financial contracts that are sensitive to *credit risk*, that is to say, the risk of an economic loss due to the failure -or *default*- of a counterpart to fulfill its contractual obligations. When a contract or a firm defaults it is said that the *default event* occurs, and the random time τ at which the default event occurs is called *default time*. A vast majority of mathematical research devoted to credit risk is concerned with modelling default times. Two methodologies have emerged in order to model the default time: the *structural approach* which dates back to Black and Scholes (cf. (6)) and Merton (cf. (7)), and the *reduced-form approach* originated with Jarrow and Turnbull (cf. (8)).

Within structural models the value of the firm of interest is assumed to have the following dynamics under the neutral probability \mathbb{P}^*

$$V_t := \exp(L_t), \quad t \in [0, T],$$

where $(L_t)_{t \in [0, T]}$ is a Lévy process. More over, credit events are triggered by movements of the firm's value $(V_t)_{t \in [0, T]}$ to some random or not-random lower threshold, sometimes called *default barrier*. For instance, taking a constant barrier K , the default time is defined as

$$\tau := \inf\{t \in [0, T] : V_t < K\},$$

where $K < V_0$, and τ is set to be ∞ if $(V_t)_{t \in [0, T]}$ does not crosses the default barrier. Notice that the default event is defined endogenously within the model, and the information available to the modeler has to be the same that the firm's manager has. More details about this approach can be found references such as (9), and (10).

In the reduced-form approach, the modeler does not have the full information that the firm manager possesses but only a subset of it, generated by the *default process* $(H_t := \mathbf{1}_{\{\tau < t\}}, t \in [0, T])$, and several other related state variables. The value of the firm's assets and its capital is not modelled at all, and credit events are specified in terms of some exogenously specified jump process. In the literature, reduced form framework has been split into two different approaches, the *Hazard Process Models* and the *Intensity-Based Models*, depending on whether the information of the default free assets is introduced or not.

In Hazard Process models, a filtration \mathbb{F} (generated by a Brownian motion) is interpreted as the information related to the default-free assets of the market. Let us denote by $\mathbb{H} := (\mathcal{H}_t)_{t \in [0, T]}$ the filtration generated by the default process $(H_t := \mathbf{1}_{\{\tau < t\}}, t \in [0, T])$. We can consider a filtration $\mathbb{G} := (\mathcal{G}_t)_{t \in [0, T]}$ encompassing the information regarding to default-free assets, as well as the information regarding the defaultable asset of interest. More specifically, we consider the progressive enlargement given by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, for every $t \in [0, T]$. It should be emphasized that τ is not necessary a stopping time with respect to \mathbb{F} , though it is a stopping time with respect to \mathbb{G} . It is well known that in order to preclude arbitrage opportunities in a default-free market, the properly discounted asset prices have to be \mathbb{F} -semimartingales. Since the full market is also assumed to be arbitrage free, these prices must be \mathbb{G} -semimartingales as well.

Although the structural and the reduced-from approaches for credit risk modelling seem to be conceptually different, efforts has been made in order to establish relationships between them. We distinguish between two main lines of work in this matter: (i) creation of a general model for credit risk modelling encompassing the two approaches; and (ii) the pass from one approach to the other by modifying the information available

to the modeler. See for instance, (11) and (12) for approach (i), and (13) for approach (ii).

A different way to relate the structural and reduced-form models may be done by considering a Kyle-Back model for insider trading (cf. (14) and (15)). Following this approach, (16) presents a model of *asymmetric information* (i.e., when different market agents possess different informations about the market) in which both approaches play a role.

2.2 Applications to insider trading in a Kyle-Back market model

A company issues a risky asset. Assume that the process $(Z_t)_{t \in [0, T]}$ models the value of the company. This process may be taken as a Brownian motion, or as a more general process that may have a drift and jumps. Three types of agents interact in the Kyle-Back market we are considering:

- The *noise traders* who trade for liquidity or hedging reasons. They observe only their own cumulative demands -modelled by Brownian motion $(B_t)_{t \in [0, T]}$, started at 0, and independent from $(Z_t)_{t \in [0, T]}$ - and whether the risky asset has defaulted or not.
- The *informed trader* who is an agent that observes continuously in time the defaultable bond prices, and knows some additional information about the risky asset (e.g., its price at some prefixed time). Let the random variable X contains the privileged. It is plausible to think that the insider does not knows exactly X but a good estimation of it, say for instance, that she knows X modified by some perturbing noise. More specifically, let the additional information until time t be given by a family of random variables $(L_s)_{s \leq t}$. We suppose that these random variables have the following structure

$$L_t = G(X, Y_t),$$

where X is an \mathcal{F}_T -measurable random variable, the process $(Y_t)_{t \in [0, T]}$ is independent of the σ -algebra \mathcal{F}_T , and $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a given measurable function. In that sense we can define the insider's filtration \mathbb{F}_I as the filtration \mathbb{F}^B enlargement by \mathbb{F}^L , that is to say, $\mathbb{F}_I := (\cap_{s > t} (\mathcal{F}_s^B \vee \mathcal{F}_s^L), t \in [0, T])$. The random variables Y_t represent the additional noise, whereas the function G specifies how the perturbation is made. One expects in general that $Y_T = 0$ and that the variance of the noise should decrease to zero as time approaches the moment at which the additional information (possessed by the insider) is released to the public.

- The *market maker* is that one market agent that observes *total order* $(R_t)_{t \in [0, T]}$ of the noise traders and the insider, and sets the price of the risky asset. Let \mathbb{F}^R denotes the minimal right continuous

and complete filtration generated by $(R_t)_{t \in [0, T]}$. Consequently, the market maker's information \mathbb{F}^M is given by the progressive filtration enlargement $(\cap_{s > t} (\mathcal{F}_s^Y \vee \sigma(\tau \wedge s)), t \in [0, T])$.

A particular case of this scenario has been seen with Proposition 16 where it is considered

$$L_t := B_T + \int_t^T \sigma_s dW_s,$$

being $(W_t)_{t \in [0, T]}$ a Brownian motion, independent of $(B_t)_{t \in [0, T]}$, and $\int_t^T \sigma_s ds < \infty$.

In (6), the authors study the case of a company issuing a defaultable bond, with face value 1, and maturity time $T = 1$. For simplicity, the interest rate is taken as zero, and the value of the company is assumed to follow a Brownian motion $(Z_t)_{t \in [0, T]}$. In turn, default is set as

$$\tau = \inf\{t \in [0, T] : Z_t = -1\}.$$

In this scenario, the privileged information is the default time τ and no perturbation is considered, so that the insider's information is given by the initial filtration enlargement $\mathbb{F}_I = (\cap_{s > t} (\mathcal{F}_s^B \vee \sigma(\tau)), t \in [0, T])$.

After properly defining the set of insider's trading strategies \mathcal{A} , and the market maker's pricing rules \mathcal{H} , the idea is to find pairs $(H, \theta) \in \mathcal{H} \times \mathcal{A}$ such that both the insider and the market maker fulfill their respective objectives. Such a pair is called an *equilibrium*. The insider's objective is to maximize her expected wealth at time T , provided that she is risk-neutral. Whereas the market maker's objective is to set a *rational* price of the risky asset in order to *clear the market*. The main result in (16) is the existence of an equilibrium (H^*, θ^*) for which the τ is a predictable stopping time under the market maker's information \mathbb{F}^M , and such that the equilibrium total order solves $(R_t^*)_{t \in [0, T]}$ the following stochastic differential equation (SDE)

$$dR_t = dB_t + \left(\frac{1}{1 + R_t} - \frac{1 - R_t}{\tau - t} \right) \mathbf{1}_{\{\tau \leq t\}} dt.$$

Aside from an equilibrium characterization lemma proved therein, the proof of this result relays on the construction of a weak solution to the SDE. As seen in Example 13, $(Z_t)_{t \in [0, T]}$ has the following \mathbb{F}_I -decomposition.

$$dZ_t = d\beta_t + \left(\frac{1}{1 + Z_t} - \frac{1 - Z_t}{\tau - t} \right) \mathbf{1}_{\{\tau \leq t\}} dt,$$

where $(\beta_t)_{t \in [0, T]}$ is a \mathbb{F}_I -Brownian motion. It can be proved that the SDE possesses a unique strong solution, and two consequences follow from this. On the one hand, $(R_t^*)_{t \in [0, T]}$ has the same law as $(Z_t)_{t \in [0, T]}$, and thus $(R_t^*)_{t \in [0, T]}$ is a Brownian motion in its own filtration. On the other hand, $\tau = \inf\{t \in [0, T] : R_t^* = -1\}$.

Hence τ is a stopping time with respect to \mathbb{F}^{R^*} , and the filtrations \mathbb{F}^{R^*} and \mathbb{F}^M coincide, so that $(R_t^*)_{t \in [0, T]}$ is an \mathbb{F}^M -Brownian motion, too.

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